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Determining the Root Locations of Systems with Real Parameter Perturbations

Shyi-Kae Yang* and Chieh-Li Chen†
National Cheng-Kung University,
Tainan, Taiwan 70101, Republic of China

Introduction

THE problem of robust stabilization and control of interval systems has received widespread interest since the introduction of Kharitonov's theorem[1,2] and several extensions.[3-5] Recently, the problem of the robust root locus has been considered by Barmish and Tempo.[6] However, their work was limited to the case where the coefficients of the characteristic polynomial are linear affine functions of physical parameters.

In this Note, an alternative algorithm for checking the zero inclusion condition of an interval system, where the coefficients of its characteristic polynomial are multilinear affine functions of physical parameters, is proposed. Use of the algorithm for determining the robust root locations is presented. Numerical examples are given to illustrate the effectiveness of the proposed approach.

Preliminary Definitions

Let the closed-loop characteristic polynomial of an interval system be described as

$$\Delta(s, q, K) = s^m + \sum_{i=0}^{m-1} \delta_i(q, K)s^i \qquad (1)$$

where K is the loop gain, $q \in Q \subset R^\ell$ denotes a vector of physical parameters with $Q = \{q | q_i \in [q_i^-, q_i^+], i = 1, \ldots, \ell\}$ and $\delta_i(q, K)$ are multilinear affine functions of q.

Definition 1: A point $\beta \in C$ located inside a convex polygon, conv(P), is said to satisfy the point inclusion condition (PIC) with respect to conv(P) and is denoted as

$$\beta \in \mathrm{conv}(P) \qquad (2)$$

where conv(P) is the convex polygon consisting of a set of points P.

Proposition 1: A point β satisfies the PIC with respect to conv(P) if and only if the area of conv(P) is equal to the area of $\mathrm{conv}(\{p_i | p_i \in P \cup \beta\})$.

If β is chosen to be the origin of the complex plane, the zero inclusion condition[6] (ZIC) can be restated as follows.

Corollary 1: The ZIC with respect to conv(P) is satisfied if and only if

$$\mathrm{area}[\mathrm{conv}(P)] = \mathrm{area}[\mathrm{conv}(\{p_i | p_i \in P \cup (0, 0)\})] \qquad (3)$$

where area[conv(·)] denotes the area of conv(·) and P represents the images of the hypercube's vertices in the physical parameter space.

The image of linear affine (LA) mappings, LA(·), has the following feature.

$$\mathrm{LA}(Q) = \mathrm{LA}\{\mathrm{conv}[\mathrm{vert}(Q)]\} = \mathrm{conv}\{\mathrm{LA}[\mathrm{vert}(Q)]\} \qquad (4)$$

where vert(Q) is the set of the vertices of Q. For the multilinear affine (MLA) mapping, MLA(·), it satisfies that

$$\mathrm{MLA}(Q) \subset \mathrm{conv}\{\mathrm{MLA}[\mathrm{vert}(Q)]\} \qquad (5)$$

Partition Q into p^{n-1} subdomains Q_i by partitioning each $q_j, j \neq k$, into p segments, i.e.,

$$Q = \bigcup_{i=1}^{p^{n-1}} Q_i$$

If the number of partition intervals increases, the MLA mapping with respect to Q becomes the union of LA mapping with respect to the subdomains Q_i. That is, if $p \to \infty$, then

$$\mathrm{MLA}(Q) = \lim_{p \to \infty} \bigcup_{i=1}^{p^{n-1}} \mathrm{conv}\{\mathrm{MLA}[\mathrm{vert}(Q_i)]\}$$
$$= \lim_{p \to \infty} \bigcup_{i=1}^{p^{n-1}} \mathrm{conv}\{\mathrm{LA}[\mathrm{vert}(Q_i)]\} \qquad (6)$$

Therefore, the image of MLA mapping of domain Q can be obtained approximately using the union of the images of LA mapping of its subdomains.

Check of Zero Inclusion Condition for the Existence of Multilinear Dependence

For $K = K^*$ and $s = s^*$, the characteristic polynomial $f(q) = \Delta(s^*, q, K^*)$ becomes a MLA mapping. Corollary 1 gives an effective method for checking the ZIC where the coefficients of characteristic polynomial are linear affine functions of physical parameters.

Following Eq. (6), the area of the image of MLA mapping can be expressed as follows:

$$\mathrm{area}[\mathrm{MLA}(Q)] = \lim_{p \to \infty} \mathrm{area}\left(\bigcup_{i=1}^{p^{n-1}} \mathrm{conv}\{\mathrm{LA}[\mathrm{vert}(Q_i)]\}\right) \qquad (7)$$

If the right-hand term converges to an acceptable tolerance for some p, say $\bar{p}$, then

$$\mathrm{area}[\mathrm{MLA}(Q)] \approx \mathrm{area}\left(\sum_{i=1}^{\bar{p}^{n-1}} \mathrm{conv}\{\mathrm{LA}[\mathrm{vert}(Q_i)]\}\right) \qquad (8)$$

The calculation procedure for the area of the image of the MLA mapping is stated as follows.

1) Calculate the area of conv{LA[vert(Q)]} as the initial value of area[MLA(Q)] denoted as $\mathrm{area}_{\mathrm{old}}$ and set $p = 2$.

2) Partition the domain Q into p^{n-1} subdomains, denoted as Q_i, $i = 1, \ldots, p^{n-1}$, by partitioning $q_j, j \neq k$.

3) Calculate the size of

$$\mathrm{area}\left(\bigcup_{i=1}^{p^{n-1}} \mathrm{conv}\{\mathrm{LA}[\mathrm{vert}(Q_i)]\}\right)$$

as the new size of area[MLA(Q)] denoted as $\mathrm{area}_{\mathrm{new}}$.

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*Postgraduate Student, Department of Mechanical Engineering.
†Associate Professor, Institute of Aeronautics and Astronautics.

4) If the relative convergence index

$$\left|\frac{\text{area}_{\text{old}} - \text{area}_{\text{new}}}{\text{area}_{\text{old}}}\right| \leq \text{acceptable tolerance} \tag{9}$$

then area[MLA(Q)] = area_{new} and stop, else $\text{area}_{\text{old}} = \text{area}_{\text{new}}$, $p = p + 1$, and go to step 2.

Example 1: Consider a bilinear polynomial

$$f(q_1,q_2) = (3 - 4q_1 - 10q_2 + 12q_1q_2) + j(7 - 10q_1 - 8q_2 + 12q_1q_2)$$

where q_1 and q_2 are both perturbed in [0,1]. Suppose the acceptable tolerance = 0.005; the size of the image of the MLA mapping corresponding to the partition interval is shown in Table 1.

The flow chart for checking the ZIC with respect to the image of a MLA mapping is shown in Fig. 1.

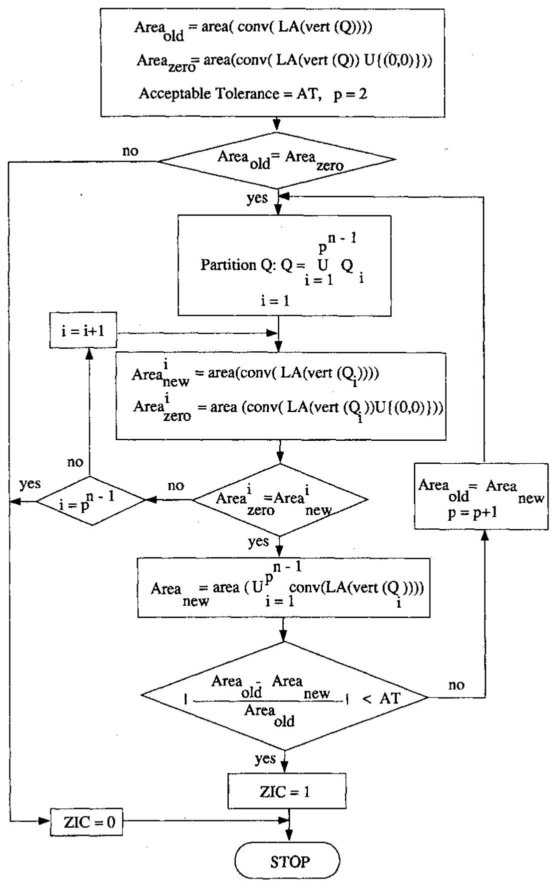

Fig. 1 Flow chart for checking the zero inclusion condition.

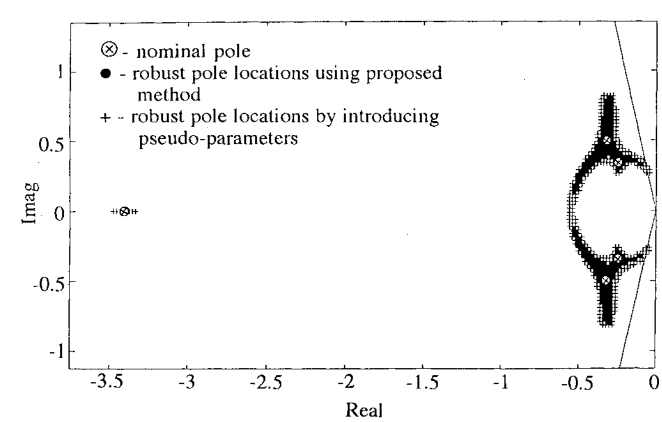

Fig. 2 Robust pole locations of example 2.

Table 1 Area of the image of MLA mapping

Number of partitions of domain Q	Area of image of union of LA mapping	$\left\lvert\frac{\text{area}_{\text{old}} - \text{area}_{\text{new}}}{\text{area}_{\text{old}}}\right\rvert$
1, ($Q = [0,1] \times [0,1]$)	36	——
2, ($Q_i = [0,1] \times [(i-1)/1, i/2]$, $i = 1 \sim 2$)	30.5785	0.1682
——	——	——
8, ($Q_i = [0,1] \times [(i-1)/8, i/8$, $i = 1 \sim 8$)	24.8595	0.066
9, ($Q_i = [0,1] \times [(i-1)/9, i/9]$, $i = 1 \sim 9$)	24.7603	0.0040

Since the perturbed set in the parameter space is a hypercube, the image is simply connected for linear and multilinear affine mapping cases. That is, the robust root region around each nominal pole is simply connected. Therefore, a search procedure[7] can be applied associated with the proposed result to improve the efficiency of determining the robust root location.

Design Example

Example 2: Consider the numerical example discussed by Biernacki et al.[8]; the plant transfer function is described as

$$P(s) = \frac{J_2s^2 + ds + k}{s^2[J_1J_2s^2 + (J_1 + J_2)ds + (J_1 + J_2)k]}$$

where $J_1 = J_{10} + q_1$, $J_2 = J_{20} + q_2$, $k = k_0 + q_3$, and $d = d_0 + q_4$ with nominal values $J_{10} = J_{20} = 1$, $k_0 = 0.245$, $d_0 = 0.0218973$, and q_1, $q \in [-0.03, 0.03]$, $q_3 \in [-0.155, 0.155]$, and $q_4 \in [-0.0181026, 0.0181026]$, respectively.

Suppose the design purpose is to guarantee a robust performance with the damping ratio $\zeta \geq 0.2$. Choose the first-order compensator as

$$C(s) = \frac{s + 0.2}{s + 4.5}$$

The design procedure is as follows.

1) To satisfy the damping ratio, K must be between 1.8 and 7.

2) Select $K^* = 4$ and determine the robust root locations around each nominal pole.

3) Figure 2 shows that the robust root locations are all interior to the desired region. Thus, the value $K^* = 4$ can be chosen as a design value for achieving the robust performance.

In this example, if the pseudoparameters are introduced to replace the multilinear terms of the coefficients of the characteristic equation, a more conservative result is shown in Fig. 2.

Conclusions

In this Note, an alternative algorithm for checking the zero inclusion condition is presented. With the proposed algorithm and using a suitable search technique, the determination of the root locus of interval systems is greatly improved for systems with the coefficients of the characteristic polynomial multilinear affine function of physical parameters. This provides a feasible approach of robust pole assignment for systems with real parameter perturbations.

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Case of Updating the Factorized Covariance Matrix

Lawrence O. Lupash*
Intermetrics, Inc.,
Huntington Beach, California 92649

I. Introduction

IN some applications, such as the Gunship AC-130U navigation system, a special type of measurement update is required from time to time, the so-called manual updates. This type of update amounts to implementing an imposed Kalman filter gain at a specified time.

The main objective is to perform the update of the covariance matrix in $U-D$ form without reconstructing and decomposing the covariance matrix.

As is well known, the covariance matrix updating process can be described by Joseph's variant (Kalman stabilized)

$$P^+ = (I-KH)P^-(I-KH)^T + KRK^T \tag{1}$$

where P^-, P^+ is the covariance matrix prior and after measurement incorporation of $n \times n$, respectively, K the Kalman gain matrix of $n \times m$, H the observation matrix of $m \times n$, and R the measurement variance matrix of $m \times m$.

When a specific value is imposed on the Kalman gain matrix K (also called the suboptimal gain), we can determine a special structure of the updated covariance matrix P^+. For example, if we consider $m=2$, $n>2$, and

$$K = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \hline \multicolumn{2}{c}{0_{n-2,2}} \end{bmatrix}, \qquad H = \left[\begin{array}{cc|c} 1 & 0 & \\ 0 & 1 & 0_{2,n-2} \end{array}\right]$$

then

$$P^+ = \begin{bmatrix} R & 0_{2,n-2} \\ 0_{2,n-2} & P^-(n-2,n-2) \end{bmatrix}$$

where $P^-(n-2,n-2)$ is the $(n-2)\times(n-2)$ lower-right block of the matrix P^-, and 0_{ij} the null matrix with i rows and j columns.

In general, the structure of the updated covariance matrix after implementing one update can be described as follows:

$$P^+ = \begin{bmatrix} P_{11}^- & 0_{p_1p_2} & P_{13}^- \\ 0_{p_2p_1} & P_{\text{imp}} & 0_{p_2p_3} \\ P_{31}^- & 0_{p_3p_2} & P_{33}^- \end{bmatrix} \tag{2}$$

where P_{11}^-, P_{13}^-, P_{31}^-, and P_{33}^- are submatrices from the covariance matrix P^- of appropriate dimensions, i.e., P^+ is obtained from P^- by substituting p_2 rows and p_2 columns as specified in Eq. (2). The matrix P_{imp}, of $p_2 \times p_2$, is symmetric positive definite.

II. Presentation of the Method

Next, we focus our attention on determining the (U^+, D^+) factors of the updated covariance matrix P^+ as a function of the (U^-, D^-) factors of the covariance matrix P^- and the $(U_{\text{imp}}, D_{\text{imp}})$ factors of the imposed submatrix P_{imp} when the structure [Eq. (2)] is assumed.

Let us consider a general covariance matrix P with the same partitions as specified in Eq. (2) and the corresponding (U,D) factors using the same type of partitions; then we have

$$P = UDU^T = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix} \begin{bmatrix} D_1 & 0 & 0 \\ 0 & D_2 & 0 \\ 0 & 0 & D_3 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}^T$$

$$= \begin{bmatrix} U_{11}D_1U_{11}^T + U_{12}D_2U_{12}^T + U_{13}D_3U_{13}^T & U_{12}D_2U_{22}^T + U_{13}D_3U_{23}^T & U_{13}D_3U_{33}^T \\ \text{symmetric} & U_{22}D_2U_{22}^T + U_{23}D_3U_{23}^T & U_{23}D_3U_{33}^T \\ \text{symmetric} & \text{symmetric} & U_{33}D_3U_{33}^T \end{bmatrix} \tag{3}$$

Substituting the $U-D$ structure specified in Eq. (3) for both P^+ and P^- in Eq. (2), and matching the corresponding blocks (in fact only the six blocks contained in the upper triangular part of the covariance matrix) we obtain the following equations:

$$U_{11}^+D_1^+U_{11}^{+T} + U_{12}^+D_2^+U_{12}^{+T} + U_{13}^+D_3^+U_{13}^{+T}$$
$$= U_{11}^-D_1^-U_{11}^{-T} + U_{12}^-D_2^-U_{12}^{-T} + U_{13}^-D_3^-U_{13}^{-T} \tag{4}$$

$$U_{12}^+D_2^+U_{22}^{+T} + U_{13}^+D_3^+U_{23}^{+T} = 0 \tag{5}$$

$$U_{22}^+D_2^+U_{22}^{+T} + U_{23}^+D_3^+U_{23}^{+T} = P_{\text{imp}} = U_{\text{imp}}D_{\text{imp}}U_{\text{imp}}^T \tag{6}$$

$$U_{13}^+D_3^+U_{33}^{+T} = U_{13}^-D_3^-U_{33}^{-T} \tag{7}$$

$$U_{23}^+D_3^+U_{33}^{+T} = 0 \tag{8}$$

$$U_{33}^+D_3^+U_{33}^{+T} = U_{33}^-D_3^-U_{33}^{-T} \tag{9}$$

From Eqs. (9), (8), and (7), respectively, it follows that we can set

$$U_{33}^+ = U_{33}^-, \qquad D_3^+ = D_3^- \tag{10}$$

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*Lead Engineer, Navigation and Guidance Department.